

# Universality and scaling limit of weakly-bound tetramers

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The occurrence of a new limit cycle in few-body physics, expressing a universal scaling function relating the binding energies of two consecutive tetramer states, is revealed, considering a renormalized zero-range two-body interaction applied to four identical bosons. The tetramer energy spectrum is obtained when adding a boson to an Efimov bound state with energy  $B_3$  in the unitary limit (for zero two-body binding, or infinite two-body scattering length). Each excited  $N$ -th tetramer energy  $B_4^{(N)}$  is shown to slide along a scaling function as a short-range four-body scale is changed, emerging from the 3+1 threshold for a universal ratio  $B_4^{(N)}/B_3 \simeq 4.6$ , which does not depend on  $N$ . The new scale can also be revealed by a resonance in the atom-trimer recombination process.

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The rich nature of quantum few-body systems interacting with short-ranged forces is not shaped only by three-body properties. Weakly-bound tetramers composed by identical bosons and their excited states have a characteristic scale, which is independent of the trimer one, for resonant pairwise interaction in the unitary limit (zero two-body binding or infinite scattering length  $a \rightarrow \pm\infty$ ). Such property can be revealed by considering the general case, not constrained by some specific strong short-range interaction. The existence of an unsuspected new limit cycle is shown, which is expressed by an universal function relating the binding energies of two consecutive tetramer states,  $B_4^{(N)}$  and  $B_4^{(N+1)}$  (where for  $N = 0$  we have the ground-state) and the corresponding three-body subsystem binding energy  $B_3$  of an Efimov state [1, 2]. We further derive that the  $N + 1$  tetramer emerges from the 3+1 threshold for a universal ratio  $B_4^{(N)}/B_3 \simeq 4.6$ , which does not depend on  $N$ . The tetramers move as the short-range four-body scale is changed. The existence of the new scale can be also revealed by a resonance in the atom-trimer recombination process. The resonant behavior arises when a tetramer becomes bound at the atom-trimer scattering threshold. Furthermore, the independent four-body scale implies in a family of Tjon lines [3] in the general case.

The findings reported here have the purpose to clarify and advance the field by recognizing the independent role of a four-body scale near a Feshbach resonance and its implications for cold atoms. This issue has been scrutinized in recent theoretical approaches presented in Refs. [4–9]. The relevance of such study is related the experimental possibilities to explore universal few-body properties with tunable two-body interactions. In this respect, it was found experimental indications on the ex-

istence of two tetramer states associated to an Efimov trimer [10], in agreement with theoretical analysis by Stecher et al. [7]. The studies presented in Ref. [7] also suggest that the only relevant scales near the Feshbach resonance are the two and three-body ones. However, as we are pointing out in the following, this is not enough in order to have a full description of the tetramer physics.

Experiments with cold atoms near the Feshbach resonance observed Efimov cycles in the position of the resonant three-body recombination as function of the scattering length (for recent reports on those findings, see Refs. [2, 11]). Furthermore, it was also found evidences for resonant four-body recombination, as discussed in [11]). Our results, leading to a new universal four-body limit cycle, show that the reported experiments and corresponding calculations have been limited to a very specific region of our theoretical predictions, suggesting that new physics related to the four-body scale are still unrevealed.

First, one should realize that, in three dimensions, the collapse of the three-body ground-state, as the two-body range  $r_0$  is reduced to zero (Thomas collapse [12]), and the accumulation of three-body excited states in the limit that the two-body scattering length  $a$  goes to infinity (Efimov effect [1]), which are recognized related effects (Thomas-Efimov effect [13, 14]), are manifestations of the sensitivity of the low-energy physics to short-range effects, which are parameterized by a three-body scale. After such scale has been established, within an appropriate scaling approach, as the one proposed in Ref. [15], it was shown that the dislocation of the three-body scale in respect to the two-body one can be clearly revealed by resuming the Efimov plot (left frame of Fig. 1) in a single curve (right frame of Fig. 1), named scaling function  $\mathcal{F}_3^{(N)}$ . This function in the zero-range limit depends on  $a = \pm 1/\sqrt{B_2}$ , positive for bound and negative for virtual dimers, in units of the natural length scale of a trimer:

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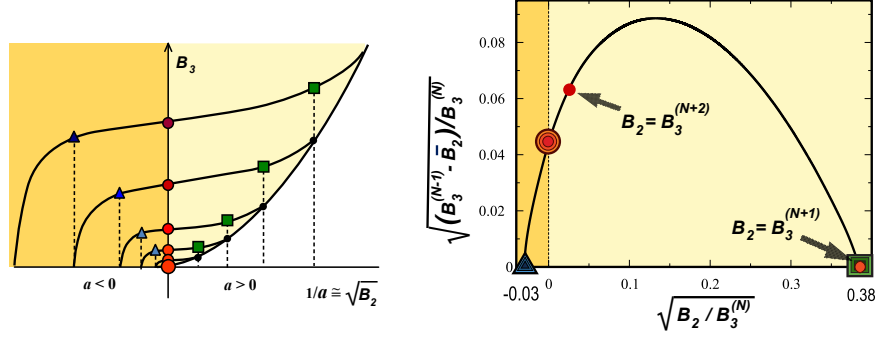


FIG. 1: Schematic Efimov plot (left) with the corresponding three-body scaling function (right), where  $B_2 = \hbar^2/(ma^2)$ , with  $m$  the atom mass (Our units are such that  $m = 1$  and  $\hbar = 1$ ). As shown, the sequence of three-body energies  $\{B_3^{(0)}, B_3^{(1)}, B_3^{(2)}, \dots\}$  can be replaced by just one exact quantitative scaling plot [15] (right). For  $B_2 = 0$ , the Efimov tower of weakly-bound  $s$ -wave states of maximum symmetry are geometrically separated by  $B_3^{(N+1)} = e^{-\frac{2\pi}{s_0}} B_3^{(N)}$  with  $s_0 = 1.00624$  (empty circles).

$$\sqrt{\frac{B_3^{(N+1)} - \bar{B}_2}{B_3^{(N)}}} \equiv \mathcal{F}_3^{(N)} \left( \pm \sqrt{\frac{B_2}{B_3^{(N)}}} \right), \quad (1)$$

The scaling function turns out to be universal and after few cycles become independent on  $N$ ; i.e., in the limit  $N \rightarrow \infty$  it reaches a renormalization-group invariant limit cycle [16, 17]. It is one among many possible model independent correlations between three-body observables for short-ranged interactions, i.e., when the range  $r_0$  goes to zero as  $|a|/r_0 \rightarrow \infty$ . Range effects can become relevant as  $|a|/r_0$  decreases (see, e.g. [18]).

The left and right frames of Fig. 1 and Eq. (1) capture the essential physics of the Efimov effect for small but non vanishing dimer energies, with three-identical particles. How this strange and curious picture changes by adding to a weakly bound three-boson system another identical boson? Does it exist a scaling function, for the ratio between two consecutive four-body binding energies  $B_4^{(N+1)}/B_4^{(N)}$ , in (or near) the limit  $a \rightarrow \pm\infty$ , having a similar role as the one found for three-boson systems? Our answer is yes! For convenience, by considering a given three-body energy  $B_3$  and  $a \rightarrow \pm\infty$ , we will write such a function in analogy with Eq. (1), such that it is zero defined for each case that the three-body binding energy  $B_3$  coincides with one of the excited four-body energy states. Therefore, the four-body scaling function will be defined as

$$\sqrt{\left[ \frac{(B_4^{(N+1)} - B_3)}{B_4^{(N)}} \right]} = \mathcal{F}_4^{(N)} \left( \sqrt{\frac{B_3}{B_4^{(N)}}} \right). \quad (2)$$

We found theoretically a simple way to move the four-body bound-state energy while keeping the three-body one unchanged. The essential idea is to recognize that the interacting three-body subsystem propagation within the

where  $\bar{B}_2 = B_2$  for bound (plus sign) two-body system and set to zero for virtual states (minus sign).

four-body system carries the three-body scale, i.e., the three-body parameter that drives the Thomas-Efimov effect. Such parameter fixes the three-body properties. The four-body scale parameterizes the physics carried by the ultraviolet momentum region of the four-boson propagation between different fully interacting three-body cluster propagation or disjoint two-body clusters propagation in the Faddeev-Yakubovsky (FY) equations (see technical details in the appendix). This is the essence of the renormalization procedure that have been developed in a previous work of our group [5].

We explore the universal physics attached to the new parameter, by solving the four-body bound-state equations, while preserving the three-body binding energies and corresponding scale unchanged, in the unitary limit. Our conclusions are supported by extensive and accurate numerical calculations of the ground and excited tetramer states energies and momentum space wave functions. We verified the sensitivity of the four-body bound-state energy to the four-body scale by moving it. We found that excited four-body states come out from the atom-trimer scattering threshold as the four-body parameter is driven to short distances or to the ultraviolet momentum region. Figure 2 illustrates this phenomenon. We see a close analogy to the left frame of Fig. 1 (for  $a > 0$ ), where the three-body excited states comes from the two-body scattering threshold as the dimer binding decreases  $|B_2| \rightarrow 0$ . The binding energies of two consecutive tetramers resume themselves into a single curve depicted in Fig. 2. We observe that the convergence with  $N$  is fast towards the limit cycle, namely the new four-boson scaling function.

The universal property of the new scaling function

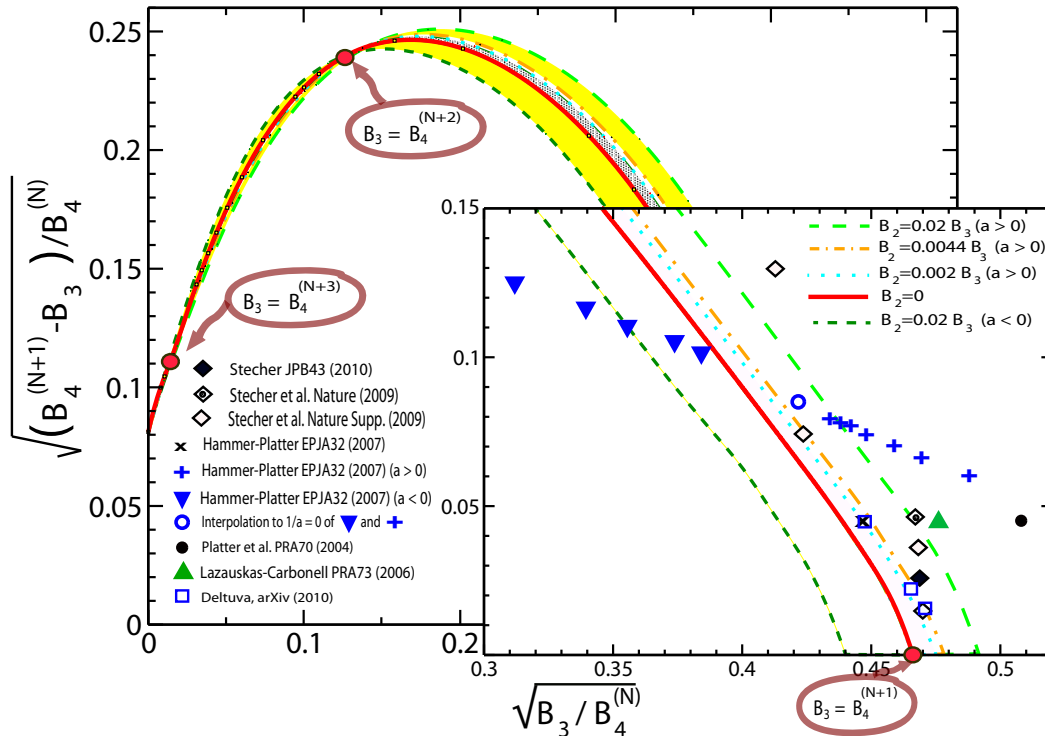


FIG. 2: Four-body scaling plot for the excited  $N + 1$  tetramer binding energy,  $B_4^{(N+1)}$ , within the renormalized zero-range model. The unitary limit is shown by the solid-red line. The virtual branch ( $a < 0$ ) is represented by  $B_2^{virtual}/B_3 = 0.02$  (small-dashed line). For non-zero  $a > 0$ , we show three cases:  $B_2/B_3 = 0.02$  (large-dashed-green line),  $B_2/B_3 = 0.0044$  (dot-dashed-orange line), and  $B_2/B_3 = 0.002$  (dotted-blue line). The symbols are indicating other model calculations. In the unitary limit, we show results of Ref. [6] ( $\times$ ), Ref. [9] (solid diamond), Ref. [19] (empty boxes), and an interpolation of the results extracted from Fig. 2 of Ref. [6] (empty-blue circle). Near the unitary limit, we show results given in Ref. [7] (dot-black and empty-black diamonds) and in Ref. [28]. From this plot, starting from right to left, by decreasing the reference three-body energy  $B_3$ , we show the first cycle starting for  $B_3 = B_4^{(N+1)}$ , when  $B_4^{(N)} \approx 4.6B_3$ , with a new cycle emerging for  $B_3 = B_4^{(N+2)}$ , when  $B_4^{(N)} \approx 64B_3$ .

can be appreciated by comparing it to different theoretical approaches in the unitary limit as shown in Fig. 2. The theoretical frameworks are very much distinct from ours. Stecher et al performed calculations with different model potentials using adiabatic hyperspherical approximation and correlated Gaussian basis set expansion [7, 9]. Hammer and Platter used one-term s-wave separable potentials within effective field theory where a repulsive three-body force stabilizes the trimer energy against collapse [6], and recently Deltuva also calculated the position of the four-body resonances with one- and two-term separable potentials [19]. Irrespectively, to which trimer the tetramers are nearest below, the energies scale according to the scaling plot shown in Fig. 2.

Actually, we are confirming the existence of an independent four-body short-range parameter as suggested in [5], with new results exploring the excited tetramer spectrum, with precise and accurate numerical results. The new scale appears not only for the relation between the first two lower tetramer states but also for the relation between other consecutive more excited states. The available results of different calculations and theoretical

methods, which are close to the unitary limit, confirm the universal scaling shown in Fig. 2. Besides the fact that such results are limited to a small region of our plot, the existence of an independent four-body scale is clearly catch. It shows the equivalence of our results with the ones found with other methods. Our theory distinguishes itself by the straightforward ability to keep fixed the three-body energies while the four-body parameter is shifted. With potential models, one should dial the two- and three-body forces maintaining the three-body states unaltered as well the two-body properties and check for the change in the binding of the tetramer states. A short-range four-body force will do the job, with fixed two- and three-body properties. Indeed, as we show through the scaling plot in Fig. 2, other theoretical approaches have embedded the four-body scale.

Calculations that have been done within other different potential models [4, 6, 7, 20] should allow a wider exploration of the ratios between three and four-body energies, in order to observe effects from the four-body independent scale. However, in the parametrization of more realistic interactions, it is obvious not so straight

to consider a fixed three-body energy, and obtain the corresponding four-body spectrum near the unitary limit.

*What does the literature say about four-body systems and Efimov effect with short-ranged interactions?*

Let us remind the past discussions on the *true* Efimov effect and universality or scaling in four-body systems. The pioneering work of Amado and Greenwood [21] have addressed the first issue and it was followed by the works of Fonseca and Adhikari [22] and Tjon and Naus [23], both performed within the Born-Oppenheimer approximation for three massive bosons and a light one. Amado and Greenwood estimated the trace of the kernel of the four-body integral equation in momentum space and showed that there is no infrared divergence from that they concluded against the true Efimov effect. However, the momentum integrals should have implicitly an ultraviolet cutoff (the four-body one) to regulate it. The trace diverges as the cutoff runs to infinity which does not conflict with the existence of an infinite number of four-boson bound states. Tjon addressed the second issue by showing that the alpha-particle and the triton binding energies are linearly correlated for different short-ranged two-nucleon potentials in a model independent way [3].

Conclusions drawn within the nuclear physics context are not general enough to assure that four-boson systems with short-ranged interactions do not have a new independent scale, as models for the nuclear interaction are pretty much constrained. Also the possibility of experiments with few-nucleon systems to explore general aspects of few-body physics due to a short-range scale are limited by the strong nuclear repulsion of the potential core. For example, it is well known that the three-body Efimov prediction on the existence of a series of low-energy three-body states, for zero two-body binding, was never undistinguished recognized for three-body nuclear systems. The most promising exploration in this aspect has been connected with the discoveries of exotic halo-nuclei systems. Finally, the Efimov effect is being recognized as a real physical phenomenon, in view of the recent cold-atom experiments performed with two-body scattering lengths varying by many decades.

Our results imply in the existence of a family of Tjon lines [3] with slopes determined by the new scale. The separation between consecutive states also depends on the four-body parameter. At the threshold to bind the excited  $N + 1$  state, we get  $B_4^{(N)} \simeq 4.6B_3$ , solution of  $\mathcal{F}_4^{(N)}\left(\sqrt{B_3/B_4^{(N)}}\right) = 0$ , which agrees, as the tetramer scaling plot in Fig. 2 shows, with the existing calculations at the unitary limit.

*What are the effects of a four-body scale in cold atoms, how to observe them?*

The existence of a four-body scale has an impact on cold atom physics where low-energy and universal properties of few-body systems are intensively explored, i.e., in systems with characteristic length scales much larger than the interaction range. Universal properties were in fact observed with trapped cold-atoms near a Feshbach

resonance, by dialing the atom-atom scattering length  $a$  over several order of magnitude. From their resonant contribution to inelastic collisions and the corresponding trap losses, the experiments have in fact confirmed both, the presence of geometrically separated Efimov trimers (see e.g. [24]), and two associated tetramers [10, 25, 26]. The four-body recombination experiments with tunable interaction, which are in agreement with some theoretical predictions [7], hopefully can also explore regions, for large  $|a|$ , where a three-body binding is much smaller than the four-body one, in order to verify the four-boson scaling behavior we are predicting in Fig. 2. Four-body observables, like four-boson recombination rates or atom-trimer or dimer-dimer scattering lengths, can exhibit correlations not constrained by one low-energy s-wave three-boson observable and  $a$ , as exemplified in Fig. 2, by the limit cycle for two consecutive tetramer states.

The coupled channel nature of the Feshbach resonance induces, by the reduction of the problem to a single channel, three and four-atom potentials, which can drive independently the corresponding physical scales. The induced two-body interaction in the open channel is attractive, producing a near-threshold S-matrix pole corresponding to a weakly-bound or virtual two-body state. The induced three-body interaction can be either attractive or repulsive, because it produces three-body amplitudes where the resonance is shared by different two-particle subsystems. For example, the interaction excites one atom of a pair in the closed channel that interacts with the third atom in the closed channel of a new pair. The magnitude of the few-atom force should increase by approaching the Feshbach resonance due to vanishing energy denominators of the virtual intermediate propagations of the subsystems in the closed channel [5]. As one cannot a priori exclude the relevance of three- and four-atom effective interactions in the open channel, one could argue why the observed positions of the four-atom resonances seem to agree with the universal theory, with no need of a new scale beyond the three-boson one [11, 26]. We suspect that those experimental results are in a region where only two-body potentials are important, with much smaller or slowly varying three-body interactions.

However a signature of many-body forces near the Feshbach resonance is emerging. The need for a dislocation of the three-body parameter was found in the recombination rates of  $^{39}\text{K}$  by Zaccanti et al. [25] and in  $^7\text{Li}$  by Pollack et al. [26] when crossing the Feshbach resonance, also in an experiment of atom-dimer loss in an ultracold trapped gas of a mixture  $^6\text{Li}$  atoms in three hyperfine states performed by Nakajima et al. [27]. These works found that the locations of the recombination peaks disagree with the predictions of the universal theory with a fixed value of the three-body parameter. In the case of  $^6\text{Li}$  even contributions from two-body nonuniversal properties were excluded, and more, it was shown that different two-body models lead to a model independent interpretation of the nonuniversal physics of the

Efimov trimers. Therefore, there are striking evidences for a nonuniversal nature of the short-range three-body physics through the variation of the three-body parameter in coldatom experiments. These experiments led us to convey that: *i*) nonuniversal short-range physics beyond two-body properties emerges near a Feshbach resonance; and *ii*) model independent predictions are still possible under the situation *i*). This is an indication of the presence of a short-range three-body force, that moves the three-body scale. Correlations between observables survives the change in the three-body parameter supporting the conclusion of *model independence* found by Nakajima et al.

Once a short range three-body force appears the very same mechanism produces effective four-body forces acting in the open channel. In the presence of three- or even four-body forces the three and four-body short range scales can move and detach the physics of trimers from tetramers. Therefore makes sense to search for effects that are tied to the different scales that parameterize the short range part of effective interaction in the open channel and are driven by different forces in trapped cold atoms near a Feshbach resonance.

In order to verify experimentally the scaling due to the four-body parameter, we suggest to tune the large and negative atom-atom scattering lengths in a region where Borromean trimers are possible without the formation of weakly-bound dimers. In this case trap losses due to inelastic dimer-dimer and atom-dimer collisions are absent. A resonant contribution to trap losses due to atom-trimer inelastic collisions occurs when the tetramer goes to the atom-trimer scattering threshold, i.e., its binding energy becomes equal to the trimer one. The position of the atom-trimer resonance is not only a function of the atom-atom scattering and the three-body scale, but it will also depends on the new four-body scale. Therefore, in this case the Efimov ratio of 22.7 between the values of the scattering length, corresponding to the position of consecutive resonances, is not assured. The ratio between the scattering lengths where two consecutive tetramers becomes unbound depends not only on the three-body parameter but also on the four-body one. The same is true for the ratio between the scattering lengths where the Borromean trimer and tetramer disappears.

The physics of four-atom systems close to a Feshbach resonance demands one three and one four-body scale which move as the large scattering length is tuned, driven by different few-body forces. As odd it can sound, universality is still shaping the physical quantities: the four-body short-range scale resumes itself in new limit cycles, which brings universal properties through scaling functions by means of correlations between two different low-energy observables of weakly-bound tetramers, as the binding energies of two consecutive tetramers. A strong experimental evidence revealing the new physics will be the observation of resonant inelastic collisions in the atom-trimer channel with large and negative scattering lengths.

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## Appendix A: Renormalized Faddeev-Yakubovsky equations

We solve the four-boson Faddeev-Yakubovsky equations in momentum space, for a zero-range potential by considering a regularizing procedure, which is based in a subtraction approach where it is introduced a renormalizing momentum scale  $\mu_4$ , such that the four-body free Green's function  $G_0(E)$  is replaced by  $G_0(E) - G_0(-\mu_4^2)$ . This approach, detailed in Ref. [5], is generalizing the subtracted equation for trimers, given in Ref. [29]:

$$\begin{aligned} \tau^{-1}(e_2^{(1,3)})|\mathcal{K}_{ij,k}^l\rangle - \mathcal{G}_{ij;ik}^{(3)}|\mathcal{K}_{ik,j}^l\rangle - \mathcal{G}_{ij;jk}^{(3)}|\mathcal{K}_{jk,i}^l\rangle = \\ = \mathcal{G}_{ij;ik}^{(4)}[|\mathcal{K}_{ik,l}^j\rangle + |\mathcal{H}_{ik,jl}\rangle] + \mathcal{G}_{ij;jk}^{(4)}[|\mathcal{K}_{jk,l}^i\rangle + |\mathcal{H}_{jk,il}\rangle], \\ \tau^{-1}(e_2^{(2,2)})|\mathcal{H}_{ij,kl}\rangle = \mathcal{G}_{ij;kl}^{(4)}[|\mathcal{K}_{kl,i}^j\rangle + |\mathcal{K}_{kl,j}^i\rangle + |\mathcal{H}_{kl,ij}\rangle], \end{aligned} \quad (\text{A1})$$

where  $e_2^{(1,3)} = E - E_{ij,k} - E_l$  and  $e_2^{(2,2)} = E - E_{ij,kl} - E_{kl}$  are the two-body subsystem energies in the 3+1 and 2+2 partitions, respectively. The projected Green's function operators are  $\mathcal{G}_{ij;ik}^{(N)} := \langle \chi_{ij} | G_0^{(N)} | \chi_{ik} \rangle$  with  $N$  equal 3 or 4, with the subtracted Green's functions given by  $G_0^{(3)} = [E - H_0]^{-1} - [-\mu_3^2 - H_0]^{-1}$  and  $G_0^{(4)} = [E - H_0]^{-1} - [-\mu_4^2 - H_0]^{-1}$ . The two-boson scattering amplitude with a proper normalization is given by  $\tau^{-1}(x) = 2\pi^2 [a^{-1} - \sqrt{-x}] \rightarrow -2\pi^2 \sqrt{-x}$  in the unitary limit and  $\langle \tilde{q}_{ij} | \chi_{ij} \rangle = 1$ . The four-boson integral equations for the reduced FY amplitudes are projected to states of total angular momentum zero. The corresponding set of homogeneous integral equations, which after discretization turns into a huge matrix eigenvalue equation, is solved by a Lanczos-like method, by iteration. This method is very efficient for few-body problems (see [30]). For the discretization of continuous momentum and angle variables we have used Gaussian-quadrature grid points with hyperbolic and linear mappings, respectively.

The two-, three- and four-body momentum scales are  $a^{-1}$ ,  $\mu_3$  and  $\mu_4$ , respectively, which give us the ground and excited tetramer binding energies, for  $a = \pm\infty$  depending on the momentum scales as  $B_4 = \mu_4^2 \mathcal{B}_4(\mu_3/\mu_4)$ . The four-body scaling function, given by Eq.(2), is obtained when  $\mu_3$  and  $\mu_4$  are replaced by  $B_3$  and  $B_4^{(N)}$ . By keeping fixed the two- and the three-body scales and moving independently the four-body scale, we obtain the

TABLE I: Tetramer ground and excited state binding energies for different four-body scales, considering the unitary limit (infinite two-body scattering length).

$\mu_4/\mu_3$	$B_4^{(0)}/B_3$	$B_4^{(1)}/B_3 - 1$	$B_4^{(2)}/B_3 - 1$
1	3.10		
1.6	4.70	$7.1 \times 10^{-4}$	
5	12.5	0.531	
10	24.6	1.44	
21	63.5	3.62	$3.2 \times 10^{-4}$
40	184	7.65	0.203
70	$5.20 \times 10^2$	12.9	0.629
100	$1.04 \times 10^3$	20.5	1.17
200	$4.06 \times 10^3$	50.8	2.86
300	$9.11 \times 10^3$	101	4.75
400	$1.62 \times 10^4$	153	6.28

tetramer binding energies for the ground and excited states.

In Table I, we have listed our numerical results for tetramer ground and excited state binding energies, considering the unitary limit (infinite two-body scattering length) for scale ratios  $\mu_4/\mu_3$  varying from 1 to

400. The results correspond to the solid-red line shown in Fig. 2. According to the results,  $\mu_4/\mu_3 \cong 1.6$  is the threshold for the first tetramer excited state, and  $\mu_4/\mu_3 \cong 21$  the threshold for the second tetramer excited state. The third tetramer excited state should emerge close to  $\mu_4/\mu_3 \approx 240$ . The binding energy ratio of two consecutive tetramers at these critical values are  $B_4^{(N)}/B_4^{(N+1)} \cong 4.6$ , which is consistent with the results obtained by other authors.

By dropping the H-channel or cutting the momentum integration below  $\mu_4$ , the dependence on the new scale for the four-boson physics will be minimized (may be even completely removed!). Therefore, unreasonable selection of the cut-off values in mapping of momentum variables and for very large  $\mu_4$  can lead to convergence in the four-body binding energies rather than collapse. So, for mapping of momentum variables one should not only consider large enough cut-off values, consistent with used four-body scale to achieve cut-off independent results, one should also consider reasonable number of mesh points close to zero momentum region. Since the iteration of coupled equations (A1) requires a very large number of multi-dimensional interpolation Yakubovsky components, we have used the Cubic Hermite Splines for its accuracy and high computational speed.

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